

## Hankel Determinants for Some Common Lattice Paths

Robert A. Sulanke and Guoce Xin

**ABSTRACT.** For a single value of  $\ell$ , let  $f(n, \ell)$  denote the number of lattice paths that use the steps  $(1, 1)$ ,  $(1, -1)$ , and  $(\ell, 0)$ , that run from  $(0, 0)$  to  $(n, 0)$ , and that never run below the horizontal axis. Equivalently,  $f(n, \ell)$  satisfies the quadratic functional equation  $F(x) = \sum_{n \geq 0} f(n, \ell)x^n = 1 + x^\ell F(x) + x^2 F(x)^2$ . Let  $H_n$  denote the  $n$  by  $n$  Hankel matrix, defined so that  $[H_n]_{i,j} = f(i+j-2, \ell)$ . Here we investigate the values of such determinants where  $\ell = 0, 1, 2, 3$ . For  $\ell = 0, 1, 2$  we are able to employ the Gessel-Viennot-Lindström method. For the case  $\ell = 3$ , the sequence of determinants forms a sequence of period 14, namely,

$$(\det(H_n))_{n \geq 1} = (1, 1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1, 1, 1, 0, 0, -1, -1, -1, \dots)$$

For this case we are able to use the continued fractions method recently introduced by Gessel and Xin. We also apply this technique to evaluate Hankel determinants for other generating functions satisfying a certain type of quadratic functional equation.

**RÉSUMÉ.** Pour une seule valeur de  $\ell$ , soit  $f(n, \ell)$  le nombre des chemins treillis que utilise les pas  $(1, 1)$ ,  $(1, -1)$ , et  $(\ell, 0)$ , vient de  $(0, 0)$  à  $(n, 0)$ , et que ne vient jamais dessous l'axis horizontale. Équivalentement, le  $f(n, \ell)$  satisfait l'équation fonctionnelle quadratique  $F(x) = \sum_{n \geq 0} f(n, \ell)x^n = 1 + x^\ell F(x) + x^2 F(x)^2$ . Soit  $H_n$  le  $n$  par  $n$  matrice de Hankel, définit pour que  $[H_n]_{i,j} = f(i+j-2, \ell)$ . Nous examinons de tels déterminants où  $\ell = 0, 1, 2, 3$ . Pour  $\ell = 0, 1, 2$  nous pouvons employer la méthode de Gessel-Viennot-Lindström. Pour le cas  $\ell = 3$ , ls séquence de déterminants forme une séquence de période 14, à savoir

$$(\det(H_n))_{n \geq 1} = (1, 1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1, 1, 1, 0, 0, -1, -1, -1, \dots)$$

Pour ce cas que nous pouvons utiliser la méthode de fractions continuées récemment introduit par Gessel et Xin. Nous appliquons aussi cette technique pour évaluer les déterminants de Hankel pour l'autres fonctions génératrices qui satisfait un certain type d'équation fonctionnelle quadratique.

### 1. Introduction

We will consider lattice paths that use the following three steps:  $U = (1, 1)$ , the up diagonal step;  $H = (\ell, 0)$ , the horizontal step of length  $\ell$ , where  $\ell$  is a single nonnegative integer; and  $D = (1, -1)$ , the down diagonal step. Further, each  $H$  step will be weighted by  $t$ , and the others by 1. The weight of a path is the product of the weights of its steps. The weight of a path set is the sum of the weights of its paths.

Let  $f(n, t, \ell)$  denote the weight of the path set of paths running from  $(0, 0)$  to  $(n, 0)$  that never run below the  $x$ -axis. When  $t = 1$ , weight becomes cardinality. For example,

- $f(n, 0, 0)$ , equivalently  $f(n, 0, \ell)$ , is the weight of a set of Dyck paths, counted by the aerated Catalan numbers:  
 $(f(0, 0, 0), f(1, 0, 0), f(2, 0, 0), \dots) = (1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, 0, 429, \dots)$ .
- $f(n, t, 1)$  is the weight of a set of Motzkin paths, counted by the Motzkin numbers:  
 $(f(0, 1, 1), f(1, 1, 1), f(2, 1, 1), \dots) = (1, 1, 2, 4, 9, 21, 51, 127, 323, 835 \dots)$ .
- $f(n, t, 2)$  is the weight of a set of large Schröder paths, counted by the aerated large Schröder numbers:  
 $(f(0, 1, 2), f(1, 1, 2), f(2, 1, 2), \dots) = (1, 0, 2, 0, 6, 0, 22, 0, 90, 0, 394, 0, \dots)$ .
- $(f(0, 1, 3), f(1, 1, 3), f(2, 1, 3), \dots) = (1, 0, 1, 1, 2, 3, 6, 10, 20, 36, 72, 136, 273, 532, \dots)$

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Previously, Pergola, *et al* [9] and Sulanke [11] have considered such generalized Motzkin paths for various values of  $\ell$  and have given additional references. Letting

$$F(x) = \sum_{n \geq 0} f(n, t, \ell) x^n$$

denote the generating function for  $f(n, t, \ell)$ , we find by a common combinatorial decomposition that  $F(x)$  satisfies the functional equation

$$F(x) = 1 + tx^\ell F(x) + x^2 F(x)^2.$$

Any sequence  $A = (a_0, a_1, a_2 \dots)$  defines a sequence of Hankel matrices,  $H_1, H_2, H_3 \dots$ , where  $H_n$  is an  $n$  by  $n$  matrix with entries  $(H_n)_{i,j} = a_{i+j-2}$ . For instance, the sequence  $(f(n, 1, 3))_{n \geq 0}$  yields

$$H_1 = [1], \quad H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 6 \end{bmatrix}$$

Our interest is to consider, for nonnegative integer  $\ell$ , the corresponding sequence of determinants  $\det(H_n)$  where each matrix  $H_n$  has entries

$$(H_n)_{i,j} = f(i+j-2, t, \ell).$$

The following propositions constitute our main results:

**PROPOSITION 1.1.** *For  $n \geq 0$ ,  $\ell = 1$ , and arbitrary  $t$  (including  $t = 0$ , yielding the Dyck path case)*

$$\det(H_n) = 1.$$

**PROPOSITION 1.2.** *For  $n \geq 0$ ,  $\ell = 2$ , and arbitrary  $t$  (including  $t = 0$ , yielding the Dyck path case),*

$$\det(H_n) = \begin{cases} (1+t)^{n^2/4} & \text{if } n \text{ is even} \\ (1+t)^{(n-1)(n+1)/4} & \text{if } n \text{ is odd} \end{cases}$$

**PROPOSITION 1.3.** *For  $t = 1$  and  $\ell = 3$ ,*

$$(\det(H_n))_{n \geq 1}^{14} = (1, 1, 0, 0, -1, -1, -1, -1, -1, 0, 0, 1, 1, 1).$$

Moreover, if  $m, n \geq 0$  with  $n - m = 0 \pmod{14}$  then  $\det(H_m) = \det(H_n)$ .

In Section 2, using the well-known combinatorial method of Gessel-Viennot-Lindström [3] [5] [13], we will prove Propositions 1.1 and 1.2. Our proof of Propositions 1.1 is essentially that of Viennot [13] who also used the method to calculate various other Hankel determinants relating to Motzkin paths. Aigner [1] also studied such determinants. We note that earlier Shapiro [10] demonstrated that the Hankel determinants for the usual Catalan numbers is 1. For the large Schröder numbers  $(r(n))_{n \geq 0} = 1, 2, 6, 22, 90, 394, \dots$  whose generating function satisfies

$$R(x) = \sum_{k \geq 0} r(k)x^k = 1 + xR(x) + xR(x)^2,$$

we show that the  $n$ -order Hankel determinant is  $2^{n(n-1)/2}$ , as stated in Proposition 2.1.

We remark that the problem of evaluating Hankel determinants corresponding to a generating function has received significant attention as considered by Wall [14]. One of the basic tools for such evaluation is the method of continued fractions, either by  $J$ -fractions in Krattenthaler [8] or Wall [14] or by  $S$ -fractions in Jones and Thron [7, Theorem 7.2]. However, both of these methods need the condition that the determinant can never be zero, a condition not always present in our study. Recently, Brualdi and Kirkland [4] used the  $J$ -fraction expansion to calculate Hankel determinants for various sequences related to the Schröder numbers. A slight modification of their proof of [4, Lemma 4.7] proves our Proposition 2.1 for  $t = 1$ .

In Section 3 we establish the periodicity of 14 for the case  $\ell = 3$  of Proposition 1.3, by the continued fraction method recently developed by Gessel and Xin [6]. In the final section, we review their technique more generally: it yields a transformation for generating functions, satisfying a certain quadratic functional equation, that also transforms the associated Hankel determinants in a simple manner. We apply this transformation to evaluate the Hankel determinants for the cases  $\ell = 1, 2$  (again) and for other path enumeration sequences related to  $\ell = 3$ .

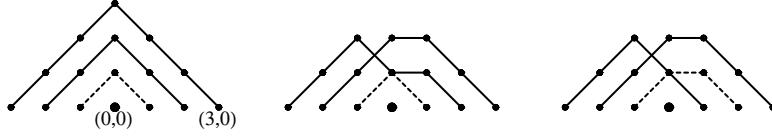


Figure 1: Some of the 4-tuples of paths for  $\ell = 1$  and for I-T-CONFIG with  $[(0, 0), (-1, 0), (-2, 0), (-3, 0)]$  and  $[(0, 0), (1, 0), (2, 0), (3, 0)]$ . In each of these 4-tuples there is a point path (a path of zero length) at  $(0, 0)$ . The first 4-tuple is the only nonintersecting 4-tuple for this case. The second and third 4-tuples are intersecting only at the point  $(0, 1)$ . The second 4-tuple corresponds to the permutation 1243 having sign of -1, while the third corresponds to the permutation 1342 having sign of 1. These two 4-tuples cancel one another under the Gessel-Viennot-Lindström method.

## 2. Employing the Gessel-Viennot-Lindström method

Assuming a rudimentary knowledge of the Gessel-Viennot-Lindström method, we reformulate it to our needs. All lattice paths use the three steps as previously defined. Given an  $n$ -tuple of lattice paths on the  $\mathbb{Z} \times \mathbb{Z}$  plane, we say that it is *nonintersecting* if no steps from different paths share a common end point. Thus an nonintersecting  $n$ -tuple may have paths crossing or touching at points other than a common step end point.

Let  $[(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)]$  and  $[(x'_1, y'_1), (x'_2, y'_2), \dots, (x'_n, y'_n)]$  denote two lists of distinct lattice points such that

$$x_{k+1} \leq x_k \leq 0 \text{ and } 0 \leq y_k \leq y_{k+1}$$

and

$$0 \leq x'_k \leq x'_{k+1} \text{ and } 0 \leq y'_k \leq y'_{k+1}.$$

We will refer to such a pair of lists as an “I-T-CONFIG” of order  $n$  as their points will be the initial and terminal points for each  $n$ -tuple of paths being considered.

Let  $P_{i,j}$  denote the set of all paths running from  $(x_i, y_i)$  to  $(x_j, y_j)$  that never run below the  $x$ -axis, with  $|P_{i,j}|$  denoting the sum of the weights of its paths. Let  $S_n$  denote the set of permutations on  $\{1, 2, 3, \dots, n\}$ . For any permutation  $\sigma \in S_n$ , let  $P_\sigma$  denote the set of all  $n$ -tuples of paths  $(p_1, p_2, \dots, p_n)$ , where  $p_i \in P_{i,\sigma(i)}$  for  $1 \leq i \leq n$ . The *signed weight* of  $(p_1, p_2, \dots, p_n) \in P_\sigma$  is defined to be  $\text{sgn}(\sigma)$  times the product of the weights of the  $n$  paths. See Figures 1 and 2.

For our purpose the Gessel-Viennot-Lindström method is formulated in a form similar to that in Viennot’s notes [13]:

**LEMMA 2.1.** *Given an I-T-CONFIG of order  $n$ , the sum of the signed weights of the nonintersecting  $n$ -tuples in  $\cup_{\sigma \in S_n} P_\sigma$  is equal to  $\det(|P_{i,j}|)_{1 \leq i, j \leq n}$ .*

**PROOF OF PROPOSITION 1.1.** (A similar proof appears in [13].) By Lemma 2.1  $\det(H_n)$  is equal to the sum of the signed weights of the nonintersecting  $n$ -tuples in  $\cup_{\sigma \in S_n} P_\sigma$  for the I-T-CONFIG where  $(x_i, y_i) = (-i + 1, 0)$  and  $(x'_i, y'_i) = (i - 1, 0)$ , for  $1 \leq i \leq n$ . Thus, for this I-T-CONFIG, we seek the nonintersecting  $n$ -tuples. First, the 1-tuple  $P_{1,1}$  contains just the point path beginning and ending at  $(0, 0)$ . Next, any nonintersecting path from  $(-i + 1, 0)$ , for  $1 < i \leq n$ , must begin with an  $U$  step, while any nonintersecting path to  $(j - 1, 0)$ , for  $1 < j \leq n$ , must end with an  $D$  step. Repeating this analysis at each integer-ordinate level  $k$ , shows the nonintersecting path from  $(-i + 1, 0)$ ,  $1 \leq i \leq k$ , is forced to be a sequence of  $U$  steps followed by a sequence of  $D$  steps; moreover, it shows that any nonintersecting path from  $(-i + 1, 0)$  to  $(j - 1, 0)$ ,  $k < i, j$ , must start with  $k$   $U$  steps and end with  $k$   $D$  steps. Inductively, each nonintersecting path is a sequence of  $U$  steps followed by a sequence of  $D$  steps. The  $n$ -tuple of such paths is the only nonintersecting  $n$ -tuple of  $\cup_{\sigma \in S_n} P_\sigma$ , and it has weight equal 1.  $\square$

We will use the following in proving Proposition 1.2:

LEMMA 2.2. *For the lattice paths that use the steps  $U$ ,  $H = (2, 0)$ , and  $D$ , that never run below the  $x$ -axis, and that have the I-T-CONFIG,*

$$(x_i, y_i) = (-2i + 2, 0) \text{ and } (x'_i, y'_i) = (2i - 2, 0)$$

*for  $1 \leq i \leq n$ , the sum of the signed weights of the nonintersecting  $n$ -tuples in  $\cup_{\sigma \in S_n} P_\sigma$  equals  $(1+t)^{n(n-1)/2}$ .*

PROOF. For  $(p_1, p_2, \dots, p_n) \in (P_{1,\sigma(1)}, P_{2,\sigma(2)}, \dots, P_{n,\sigma(n)})$ , suppose that  $(p_1, p_2, \dots, p_n)$  is a nonintersecting  $n$ -tuple of paths for some permutation  $\sigma$ . Since the points in the I-T-CONFIG are spaced two units apart, the horizontal distance at any integer ordinate between any two paths of  $(P_{1,\sigma(1)}, P_{2,\sigma(2)}, \dots, P_{n,\sigma(n)})$  must be even. It follows inductively that, for  $1 \leq i \leq n$ , any path of the path set  $P_{i,\sigma(i)}$  must begin with a sequence of  $i-1$   $U$ -steps and finish with a sequence of  $\sigma(i)-1$   $D$ -steps. Thus, computing the weight of the nonintersecting  $n$ -tuples is equivalent to computing the weight of the nonintersecting  $n$ -tuples for the new (“V” shaped) initial-terminal configuration, denoted by I-T-CONFIG-NEW, defined by

$$(x_i, y_i) = (-i+1, i-1) \text{ and } (x'_i, y'_i) = (i-1, i-1)$$

for  $1 \leq i \leq n$ .

Before continuing, we notice, for example when  $t = 1$  and  $n = 4$ , that the matrix  $M(0)$  define by  $(M(0)_{i,j})_{1 \leq i,j \leq 4} = (|P'_{i,j}|)_{1 \leq i,j \leq 4}$  for I-T-CONFIG-NEW is an array of Delannoy numbers. (See [2], [12].) When  $t = 0$ ,  $M(0)$  is the initial array from Pascal’s triangle. In the following array for  $t = 1$ , the entries count the ways a chess king can move from the north-west corner if it uses only east, south, or south-east steps. Momentarily we will see the role of the argument 0 in  $M(0)$ .

$$M(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 5 & 13 & 25 \\ 1 & 7 & 25 & 63 \end{bmatrix}$$

Now for arbitrary  $t$  and  $n$ , let  $M(0)$  be the  $n$  by  $n$  matrix defined recursively by

$$M(0)_{i,j} = M(0)_{i-1,j} + tM(0)_{i-1,j-1} + M(0)_{i,j-1}$$

for  $1 < i$  and  $1 < j$  with  $M(0)_{1,j} = 1$  and  $M(0)_{i,1} = 1$  for  $1 \leq i$  and  $1 \leq j$ . By Lemma 2.1  $M(0) = (|P_{ij}|)_{1 \leq i,j \leq n}$  for I-T-CONFIGNEW. Thus  $\det(M(0))$  is equal to the weight of the nonintersecting  $n$ -tuples for I-T-CONFIG. The proof is completed once we show

$$\det(M(0)) = (1+t)^{n(n-1)/2}.$$

Given  $M(0)$ , we recursively define a sequence of  $n$  by  $n$  matrices

$$M(0), M(1), M(2), \dots, M(n-1)$$

where, for  $1 \leq k \leq n-1$ ,

$$M(k)_{ij} = \begin{cases} M(k-1)_{i,j} & \text{for } 1 \leq i \leq k \\ M(k-1)_{i,j} - M(k-1)_{i-1,j} & \text{for } k+1 \leq i \leq n \end{cases}$$

With CLAIM( $k$ ) denoting the claim that

$$\begin{aligned} M(k)_{i,j} &= M(k)_{i-1,j} + tM(k)_{i-1,j-1} + M(k)_{i,j-1} \text{ for } i, j > k, \\ M(k)_{i,i} &= (1+t)^{i-1} \text{ for } i \leq k, \\ M(k)_{i,j} &= 0 \text{ for } i > j \text{ and } j \leq k, \\ M(k)_{i,k+1} &= (1+t)^k \text{ for } i \geq k+1, \end{aligned}$$

one can establish CLAIM( $k$ ) for  $1 \leq k \leq n-1$  by induction. Since  $M(n-1)$  is upper triangular, we observe that

$$\det(M(n-1)) = (1+t)^{n(n-1)/2}.$$

By the type of row operations used to obtain the sequence  $M(0), M(1), M(2), \dots, M(n-1)$ , their determinants are equal.  $\square$

Since, by the I-T-CONFIG of Lemma 2.2,  $(H)_{i,j} = |P_{i,j}|$  counts the large Schröder paths from  $(0, 0)$  to  $(2i+2j-2, 0)$ , immediately we have the the following corollary for the Hankel determinants of the weighted non-aerated Schröder numbers:

PROPOSITION 2.1. Let  $f_n$  denote the weight of the path set of paths from  $(0, 0)$  to  $(2n, 0)$  which never run beneath the  $x$ -axis and where  $H = (2, 0)$  is weighted by  $t$ . Equivalently, let  $f_n$  satisfy

$$F(x) = \sum_{n \geq 0} f_n x^n = 1 + txF(x) + xF(x)^2.$$

Then the determinant of the  $n$ -th order Hankel matrix equals  $(1+t)^{n(n-1)/2}$ .

As a second corollary to Lemma 2.2, we have

LEMMA 2.3. For the lattice paths that use the steps  $U$ ,  $H = (2, 0)$ , and  $D$ , that never run below the  $x$ -axis, and that have the I-T-CONFIG with

$$(x_i, y_i) = (-2i+1, 0) \text{ and } (x'_i, y'_i) = (2i-1, 0)$$

for  $1 \leq i \leq n$ , the sum of the signed weights for the nonintersecting  $n$ -tuples in  $\cup_{\sigma \in S_n} P_\sigma$  is  $(1+t)^{n(n+1)/2}$ .

PROOF OF LEMMA 2.3. We first translate all paths upwards one unit and then prepend a  $U$ -step and append a  $D$ -step to every path. Next we add the point path at  $(0, 0)$ . The sum of the signed weights of the nonintersecting  $n$ -tuples in the original configuration equals that of the nonintersecting  $n+1$ -tuples in this new configuration, which in turn is given by Lemma 2.2.  $\square$

PROOF OF PROPOSITION 1.2. Suppose that  $n$  is even; the proof when  $n$  is odd is similar. Here the Hankel matrix  $(|P_{i,j}|)_{1 \leq i,j \leq n}$  corresponds to the I-T-CONFIG with

$$(x_i, y_i) = (-i+1, 0) \text{ and } (x'_i, y'_i) = (i-1, 0) \text{ for } 1 \leq i \leq n.$$

Since  $\ell = 2$ , no endpoint of a step on a path that originates from an oddly indexed initial point (i.e., a point  $(-i+1, 0)$  for odd  $i$ ) will intersect an endpoint of a step on a path that originates from an evenly indexed initial point. Moreover, for any permutation  $\sigma$  corresponding to a nonintersecting  $n$ -tuple,  $\sigma(i) - i$  must be even for each  $i$ , and hence  $\text{sgn}(\sigma) = 1$ . Thus the weight of the nonintersecting  $n$ -tuples is the product of the weight of those originating from oddly indexed initial points times the weight of those originating from evenly indexed initial points.

Hence, with  $m = n/2$ , let I-T-CONFIGA have

$$(x_i, y_i) = (-2i+2, 0) \text{ and } (x'_i, y'_i) = (2i-2, 0) \text{ for } 1 \leq i \leq m,$$

and let I-T-CONFIGB have

$$(x_i, y_i) = (-2i+1, 0) \text{ and } (x'_i, y'_i) = (2i-1, 0) \text{ for } 1 \leq i \leq m.$$

Applying Lemmas 2.2 and 2.3 to these configurations yields the weight of nonintersecting  $n$ -tuples of the original configuration as

$$(1+t)^{m(m-1)/2}(1+t)^{m(m+1)/2} = (1+t)^{m^2} = (1+t)^{n^2/4}.$$

$\square$

Next we consider Hankel determinants for sequences of path weights that ignore the initial term. For the sequence  $f(1, t, \ell), f(2, t, \ell), \dots$ , we will let  $H_n^1$  denote the matrix where the entries satisfy  $(H_n^1)_{i,j} = f(i+j-1, t, \ell)$ . See Figure 2.

PROPOSITION 2.2. For  $\ell = 1$  (Motzkin case again), the sequence of determinants satisfies the recurrence

$$\det(H_n^1) = t \det(H_{n-1}^1) - \det(H_{n-2}^1)$$

subject to  $\det(H_1^1) = t$  and  $\det(H_2^1) = (t-1)(t+1)$ .

PROOF. Aigner [1] considered the case for  $t = 1$ . For arbitrary  $t$ , our proof considers how the particular paths must look in the nonintersecting case. Observe that  $\det(H_n^1)$  is the sum of the weights of the nonintersecting  $n$ -tuples for the I-T-CONFIG( $n$ ) taken as

$$[(0, 0), (-1, 0), \dots, (n-1, 0)] \text{ and } [(1, 0), (2, 0), \dots, (n, 0)].$$

Each of these nonintersecting  $n$ -tuples belongs to one of two types: (1) those containing the path from  $(0, 0)$  to  $(1, 0)$  with all other paths forced to begin with  $U$ , end with  $D$ , and have ordinate at least one elsewhere;

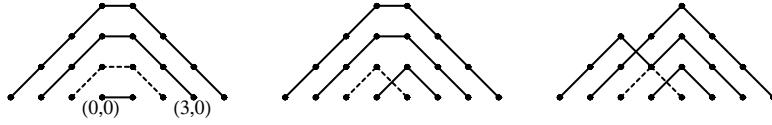


Figure 2: Three of the 4-tuples of paths for  $\ell = 1$  and for I-T-CONFIG with  $[(0, 0), (-1, 0), (-2, 0), (-3, 0)]$  and  $[(1, 0), (2, 0), (3, 0), (4, 0)]$ . The first and second 4-tuples are both nonintersecting. The first has a signed weight of  $t^4$  while the second has a signed weight of  $-t^2$ . The third is intersecting only at the point  $(0, 1)$ .

(2) those containing the path  $UD$  from  $(0, 0)$  to  $(2, 0)$  and the path  $UD$   $(-1, 0)$  to  $(1, 0)$  with all other paths forced to begin with  $UU$ , end with  $DD$ , and have ordinate at least two elsewhere. The set of the first type has a total weight  $t$  times the sum of the weights of the nonintersecting  $(n-1)$ -tuples on the I-T-CONFIG( $n-1$ ), which is  $t \det(H_{n-1}^1)$ . Since each  $n$ -tuple of the second type has the defined crossing of the path from  $(0, 0)$  with that from  $(-1, 0)$ , the set has total weight is the sign of the corresponding permutation times the sum of the weights of the nonintersecting  $(n-2)$ -tuples on the I-T-CONFIG( $n-2$ ), which is  $-\det(H_{n-2}^1)$ .  $\square$

For  $\ell = 2$ , we will indicate how Lemma 2.3 proves

**PROPOSITION 2.3.** *For  $n \geq 0$ ,  $\ell = 2$ , and arbitrary  $t$ , the sequence of determinants satisfies*

$$\det(H_n^1) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}(1+t)^{n(n+2)/4} & \text{if } n \text{ is even} \end{cases}$$

**PROOF.** Here the Hankel matrix can correspond to I-T-CONFIG with

$$(x_i, y_i) = (-i+1, 0) \text{ and } (x'_i, y'_i) = (i, 0) \text{ for } 1 \leq i \leq n.$$

Since  $\ell = 2$ , if there is a path from  $(x_i, y_i)$  to  $(x'_j, y'_j)$ , then  $i - j$  is odd. It follows that, if  $n$  is odd, there can be no  $n$ -tuples of paths for the configuration. If  $n$  is even and  $m = n/2$ , the sign of any permutation for an nonintersecting  $n$ -tuples can be shown to be  $(-1)^m$ . Thus the weight of the nonintersecting  $n$ -tuples is  $(-1)^m$  times the weight of those originating from oddly indexed initial points times the weight of those originating from evenly indexed initial points. The proof is completed by applying 2.3 to I-T-CONFIGA with

$$(x_i, y_i) = (-2i+2, 0) \text{ and } (x'_i, y'_i) = (2i, 0) \text{ for } 1 \leq i \leq m,$$

and to I-T-CONFIGB with

$$(x_i, y_i) = (-2i+1, 0) \text{ and } (x'_i, y'_i) = (2i-1, 0) \text{ for } 1 \leq i \leq m.$$

$\square$

### 3. Periodicity fourteen and continued fractions

Here we will repeatedly apply the “continued fractions method” recently developed by Gessel and Xin [6] to determine the periodicity of the sequence of Hankel determinants for  $\ell = 3$  and  $t = 1$ . This method, presented more formally in the next section, transforms both generating functions and corresponding determinants. In this section we will concentrate on the specific generating function  $F(x)$  satisfying

$$F(x) = 1 + x^3 F(x) + x^2 F(x)^2.$$

From this functional equation, or from the related recurrence for its coefficients, there appears to be no clue why the associated sequence of Hankel determinants should have a period of 14.

For an arbitrary generating function  $D(x, y) = \sum_{i,j=0}^{\infty} d_{i,j} x^i y^j$ , let  $[D(x, y)]_n$  denote the  $n$  by  $n$  determinant  $\det((d_{i,j})_{0 \leq i, j \leq n-1})$ . For any  $A(x) = \sum_{n \geq 0} a_n x^n$ , define the *Hankel matrix for A of order n*,  $n \geq 1$ , by  $H_n(A) = (a_{i+j-2})_{1 \leq i, j \leq n}$ . It is straight forward to show that the Hankel determinant  $\det(H_n(A))$  can be expressed as

$$\det(H_n(A)) = \left[ \frac{x A(x) - y A(y)}{x - y} \right]_n.$$

We will use an easily-proven “product rule” of [6] for transforming the generating functions: *If  $u(x)$  is a formal power series with  $u(0) = 1$ , then*

$$[u(x)D(x,y)]_n = [D(x,y)]_n = [u(y)D(x,y)]_n.$$

We will make five transformations showing, for  $n \geq 8$ ,

$$\det(H_n(F)) = \det \left( \text{diag} \left( [1], [1], \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}, [1], [1], H_{n-7}(F) \right) \right),$$

where the right side is the determinant of a block-diagonal matrix consisting of six blocks along the diagonal, four of which are 1 by 1 identity matrices, and having entry 0 elsewhere. It then follows that  $\det(H_n(F)) = -\det(H_{n-7}(F))$ . This implies that the period for  $\det(H_n(F))$  is 14, and Proposition 1.3 will be proved.

We start with  $F_0(x) = F(x)$ , and define  $F_i(x)$  from  $F_{i-1}(x)$  according to a transformation where each Hankel determinant for  $F_i(x)$  are derived from one for  $F_{i-1}(x)$  with the aid of the product rule, which is not always mentioned. In the following,  $F_i(x)$  will always satisfy a quadratic functional equation

$$a(x)F_i(x)^2 + b(x)F_i(x) + c(x) = 0,$$

which is equivalent to the continued fraction

$$F_i(x) = \frac{-c(x)}{b(x) + a(x)F_i(x)}.$$

In particular, for  $\ell = 3$ ,

$$F_0(x) = \frac{1}{1 - x^3 - x^2F_0(x)}.$$

**Transformation 1:** Using this continued fraction of  $F_0$ , substitution, and simplification we obtain

$$\det(H_n(F)) = \left[ \frac{xF_0(x) - yF_0(y)}{x - y} \right]_n = \left[ \frac{-xy^2F_0(y) + yx^2F_0(x) + (x - y)(yx^2 + xy^2 + 1)}{(1 - x^3 - x^2F_0(x))(1 - y^3 - y^2F_0(y))(x - y)} \right]_n.$$

Multiplying by  $(1 - x^3 - x^2F_0(x))(1 - y^3 - y^2F_0(y))$ , which will not affect the value of the determinant by the product rule, we can write the determinant as

$$\left[ 1 + xy \frac{xF_1(x) - yF_1(y)}{x - y} \right]_n$$

where

$$(3.1) \quad F_1(x) = F_0(x) + x.$$

The associated matrix is block-diagonal with two blocks: the matrix [1] and the Hankel matrix for  $F_1(x)$ . Certainly,

$$\det(H_n(F_0)) = \det(H_{n-1}(F_1)).$$

From (3.1) and the functional equation for  $F_0(x)$ , we obtain the functional equation

$$F_1(x) = \frac{1 + x}{1 + x^3 - x^2F_1(x)}.$$

**Transformation 2:** Using this continued fraction for  $F_1$ , substituting in  $\frac{xF_1(x) - yF_1(y)}{x - y}$ , and multiplying by  $(1 + x^3 - x^2F_1(x))(1 + y^3 - y^2F_1(y))$  yields

$$\left[ \frac{xF_1(x) - yF_1(y)}{x - y} \right]_n = \left[ \frac{-xy^2(x + 1)F_1(y) + yx^2(y + 1)F_1(x) - (y + 1)(x + 1)(xy - 1)(x - y)}{x - y} \right]_n.$$

Upon multiplying by  $(1 + x)^{-1}(1 + y)^{-1}$ , the determinant is equal to

$$\left[ 1 + xy \frac{xF_2(x) - yF_2(y)}{x - y} \right]_n,$$

where

$$(3.2) \quad F_2(x) = F_1(x)/(1 + x) - 1.$$

The associated matrix being block diagonal shows

$$\det(H_{n-1}(F_1)) = \det(H_{n-2}(F_2)).$$

From (3.2) and the functional equation for  $F_1(x)$ , we obtain

$$F_2(x) = \frac{x^2}{1 - 2x^2 - x^3 - (x^3 + x^2) F_2(x)}.$$

**Transformation 3:** Substituting for  $F_2$  with the above fraction, simplifying, and multiplying by  $(1 + x)(1 - x - x^2 - x^2 F_2(x))(1 + y)(1 - y - y^2 - y^2 F_2(y))$  shows that the determinant  $\left[ \frac{xF_2(x) - yF_2(y)}{x - y} \right]_n$  equals

$$\left[ \frac{y^2 x^3 (y + 1) F_2(y) - x^2 y^3 (x + 1) F_2(x) - (x - y) (2 y^2 x^2 - x^2 - xy - y^2)}{x - y} \right]_n$$

which can be rewritten as

$$\left[ x^2 + xy + y^2 - 2x^2 y^2 + x^3 y^3 \frac{xF_3(x) - yF_3(y)}{x - y} \right]_n,$$

where  $F_3(x)$  is indeed a power series satisfying

$$(3.3) \quad F_3(x) = (x + 1)F_2(x)/x^2.$$

This time the corresponding matrix is a block-diagonal matrix with the block  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$  followed by the Hankel matrix for  $F_3(x)$ . Hence

$$\det(H_{n-2}(F_2)) = -\det(H_{n-5}(F_3)).$$

From (3.3) and the functional equation for  $F_2(x)$ , we obtain

$$F_3(x) = \frac{1 + x}{1 - 2x^2 - x^3 - x^4 F_3(x)}.$$

**Transformation 4:** Substituting for  $F_3$  with the fraction, simplifying, and multiplying by  $(1 - 2x^2 - x^3 - x^4 F_3(x))((1 - 2y^2 - y^3 - y^4 F_3(y))$  the determinant  $\left[ \frac{xF_3(x) - yF_3(y)}{x - y} \right]_n$  equals

$$\left[ \frac{-xy^4 (x + 1) F_3(y) + yx^4 (y + 1) F_3(x) + (y + 1) (x + 1) (xy + 1) (x - y)}{x - y} \right]_n.$$

By multiplying the generating function by  $(1 + x)^{-1}(1 + y)^{-1}$ , this determinant becomes

$$\left[ 1 + xy \frac{xF_4(x) - yF_4(y)}{x - y} \right]_n,$$

where

$$(3.4) \quad F_4(x) = 1 + x^2 F_3(x)/(1 + x).$$

Therefore,

$$\det(H_{n-5}(F_3)) = \det(H_{n-6}(F_4)).$$

From (3.4) and the functional equation for  $F_3(x)$ , we obtain

$$F_4(x) = \frac{1}{1 + x^3 - (x^3 + x^2) F_4(x)}.$$

**Transformation 5:** Substituting for  $F_4$  with the above fraction, simplifying, and multiplying by  $(1 - x^2 F_4(x))(1 - y^2 F_4(y))$  the determinant  $\left[ \frac{xF_4(x) - yF_4(y)}{x - y} \right]_n$  equals

$$\left[ \frac{-xy^2 (y + 1) F_4(y) + x^2 y (x + 1) F_4(x) - (x - y) (yx^2 + xy^2 - 1)}{x - y} \right]_n = \left[ 1 + xy \frac{xF_5(x) - yF_5(y)}{x - y} \right]_n,$$

where  $F_5(x) = (1 + x)F_4(x) - x$ . Hence,  $\det(H_{n-6}(F_4)) = \det(H_{n-7}(F_5))$ .

Finally, it is routinely checked that  $F_5(x) = F_0(x)$ .

#### 4. The quadratic transformation for Hankel determinants

One can use the method introduced in the previous section to evaluate the Hankel determinants for generating functions satisfying a certain type of quadratic functional equation. The generating functions  $F(x)$  in this section are the unique solution of a quadratic functional equation satisfying

$$(4.1) \quad F(x) = \frac{x^d}{u(x) + x^k v(x) F(x)},$$

where  $u(x)$  and  $v(x)$  are rational power series with nonzero constants,  $d$  is a nonnegative integer, and  $k$  is a positive integer. Note that if  $k = 0$ ,  $F(x)$  is not unique. Our task now is to derive a transformation  $\mathcal{T}$  so that  $\det(H_n(F)) = a \det(H_{n-d-1}(\mathcal{T}(F)))$  for some value  $a$  and nonnegative integer  $d$ . In addition to Hankel matrices for the power series  $A = \sum_{n \geq 0} a_i x^i$ , we will consider *shifted Hankel matrices*:  $H_n^k(A)$  denotes the matrix  $(a_{i+j+k-2})_{1 \leq i,j \leq n}$ . Shifted matrices have appeared in Proposition 2.2 and 2.3.

The first proposition is elementary:

**PROPOSITION 4.1.** *If  $F$  satisfies (4.1), then  $G = u(0)F$  satisfies*

$$\det(H_n(G)) = u(0)^n \det(H_n(F)), \text{ and } G(x) = \frac{x^d}{u(0)^{-1}u(x) + x^k u(0)^{-2}v(x)G(x)}.$$

**PROPOSITION 4.2.** *Suppose  $F$  satisfies (4.1) with  $u(0) = 1$ . We separate  $u(x)$  uniquely as  $u(x) = u_L(x) + x^{d+2}u_H(x)$ , where  $u_L(x)$  is a polynomial of degree at most  $d+1$  and  $u_H(x)$  is a power series.*

(i) *If  $k = 1$ , then there is a unique  $G$  such that*

$$G(x) = \frac{-v(x) - xu_L(x)u_H(x)}{u_L(x) - x^{d+2}u_H(x) - x^{d+1}G(x)},$$

*Moreover,*

$$G(x) = -xu_H(x) - x^{-d}v(x)F(x)$$

*and a shifted matrices appears with*

$$\det(H_{n-d-1}^1(G(x))) = (-1)^{d(d+1)/2} \det(H_n(F(x))).$$

(ii) *If  $k \geq 2$ , then there is a unique  $G$  such that*

$$G(x) = \frac{-x^{k-2}v(x) - u_L(x)u_H(x)}{u_L(x) - x^{d+2}u_H(x) - x^{d+2}G(x)},$$

*Moreover,*

$$G(x) = -u_H(x) - x^{k-d-2}v(x)F(x)$$

*and*

$$\det(H_{n-d-1}(G(x))) = (-1)^{d(d+1)/2} \det(H_n(F(x))).$$

**PROOF.** We prove only part (ii) as part (i) is similar. The generating function for  $H_n(F)$  is given by

$$\begin{aligned} \frac{xF(x) - yF(y)}{x - y} &= \frac{1}{x - y} \left( \frac{x^{d+1}}{u(x) + x^k v(x) F(x)} - \frac{y^{d+1}}{u(y) + y^k v(y) F(y)} \right) \\ &= \frac{-y^{d+1}u(x) - y^{d+1}x^k v(x) F(x) + x^{d+1}u(y) + x^{d+1}y^k v(y) F(y)}{(u(x) + x^k v(x) F(x))(u(y) + y^k v(y) F(y))(x - y)} \end{aligned}$$

We can multiply by  $(u(x) + x^k v(x) F(x))$  and by  $(u(y) + y^k v(y) F(y))$  without changing the above determinant by the product rule. Next we observe that  $x^d$  divides  $F(x)$ , and write  $u(x) = u_L(x) + x^{d+2}u_H(x)$  as in the proposition. The resulting generating function can be written as

$$\frac{-y^{d+1}u_L(x) + x^{d+1}u_L(y)}{x - y} + (xy)^{d+1} \frac{-x(u_H(x) - x^{k-d-2}v(x)F(x)) + y(u_H(y) + y^{k-d-2}v(y)F(y))}{x - y}.$$

We now set  $G(x) = -u_H(x) - x^{k-d-2}v(x)F(x)$ , which can be straightforwardly shown to agree with the defining functional equations. Suppose that  $u_L(x) = 1 + a_1x + \dots + a_{d+1}x^{d+1}$ , then  $\left[ \frac{xF(x) - yF(y)}{x - y} \right]_n$  is equal to the determinant of the block-diagonal matrix

$$\text{diag} \left( \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \vdots & a_{d-1} \\ 1 & a_1 & \cdots & a_d \end{pmatrix}, H(G(x)) \right).$$

The determinant of the first block is easily seen to be  $(-1)^{d(d+1)/2}$ .  $\square$

Given these propositions and that  $H^1(A) = H(x^{-1}(A(x) - A(0)))$  for any series  $A$ , we can now define our transformation  $\mathcal{T}(F)$ : *For  $F$  satisfying (4.1),*

- if  $u(0) \neq 1$ , then  $\mathcal{T}(F) = G$ , as given in Proposition 4.1.
- if  $u(0) = 1$  and  $k = 1$ , then  $\mathcal{T}(F) = x^{-1}(G(x) - G(0))$ , with  $G$  given in Proposition 4.2(i).
- if  $u(0) = 1$  and  $k \geq 2$ , then  $\mathcal{T}(F) = G$ , as given in Proposition 4.2(ii).

Moreover, the relation between  $\det(H_n(F))$  and  $\det(H_n(\mathcal{T}(F)))$  is given in Propositions 4.1 and 4.2.

**Example 1:** OTHER PROOFS OF PROPOSITIONS 1.1 AND 2.2. For Motzkin paths with arbitrary  $t$ , the generating function  $F(x)$  satisfies

$$F(x) = \frac{1}{1 - tx - x^2F(x)}.$$

Applying Proposition 4.2 so  $F_1 = \mathcal{T}(F)$  gives

$$\det(H_{n-1}(F_1)) = \det(H_n(F)) \text{ where } F_1(x) = \frac{1}{1 - tx - x^2F_1(x)}.$$

Hence,  $F(x) = F_1(x)$ , and consequently  $\det(H_n(F(x))) = 1$  for all  $n$ .

Whereas the Gessel-Viennot-Lindström method leads to a proof in the shifted case for arbitrary  $t$ , as in Proposition 2.2, we have been able to use the continued fractions technique only for  $t = 1$  and  $t = 2$ .

For  $t = 1$  we will show that  $(\det(H_n^1(F)))_{n \geq 1} = (1, 0, -1, -1, 0, 1, 1, \dots)$ , continuing with period 6. Let  $G_1(x) = (F(x) - 1)/x$ , so that  $\det(H_n^1(F)) = \det(H_n(G_1))$ . Let  $G_2 = \mathcal{T}(G_1)$  and  $G_3 = \mathcal{T}(G_2)$ , both under Proposition 4.2(ii). Since

$$G_1(x) = \frac{1+x}{1-x-2x^2-x^3G_1(x)}$$

with  $d = 0$ ,  $k = 3$ ,  $u(x) = u_L(x) = 1 - 2x$ ,  $u_H = 0$ , and  $v(x) = -(1+x)^{-1}$ , we find that

$$G_2(x) = \frac{x}{1-x-2x^2-x^2(1+x)G_2(x)}$$

with  $d = 1$ ,  $k = 2$ ,  $u(x) = u_L(x) = 1 - x - 2x^2$ ,  $u_H = 0$ , and  $v(x) = -(1+x)$ . Applying Proposition 4.2(ii) shows

$$\begin{aligned} G_3(x) &= -x^{-1}(-(1+x))G_2(x) \\ &= -x^{-1}(-(1+x))(-x(-(1+x)^{-1}))G_1(x) \\ &= G_1(x) \end{aligned}$$

and  $\det(H_{n-3}(G_3)) = -\det(H_{n-1}(G_2)) = -\det(H_n(G_1))$ , which yields the periodicity of the sequence of determinants.

For  $t = 2$  we will show that  $\det(H_n^1(F)) = n + 1$  for  $n \geq 1$ . Define,  $G_1$  to satisfy,

$$G_1(x) = \frac{2+x}{1-2x-2x^2-x^3G_1(x)}.$$

One can easily see that  $G_1(x) = (F(x) - 1)/x$  with  $G_1(0) = u_1(0)^{-1} = \det(H_1(G_1)) = \det(H_1^1(F)) = 2$ . For  $n \geq 2$ , define,  $G_n$  to satisfy,

$$G_n(x) = \frac{(n-1)^2(n^2+n+x)}{(n^2-n)(n^2-2n^2x-2x^2)-n^2(n^2-n+x)x^2G_n(x)}.$$

By induction one can show that  $G_n = \mathcal{T} \circ \mathcal{T}(G_{n-1})$  (under Prop. 4.1 then under Prop. 4.2), and that  $G_n(0) = u_n(0)^{-1} = (n-1)(n+1)/n^2$ . Also by induction and Proposition 4.1, for  $n \geq 2$ ,

$$\begin{aligned} \det(H_n(G_1)) &= \left[ 2^n \prod_{i=2}^{n-1} \left( \frac{(i-1)(i+1)}{i^2} \right)^{n+1-i} \right] \det(H_1(G_n)) \\ &= 2^n \prod_{i=2}^n \left( \frac{(i-1)(i+1)}{i^2} \right)^{n+1-i} \end{aligned}$$

which simplifies to  $\det(H_n(G_1)) = n+1$ .

**Example 2:** ANOTHER PROOF OF PROPOSITION 1.2. For large Schröder paths arbitrary  $t$ , we have

$$F(x) = \frac{1}{1-tx^2-x^2F(x)}.$$

Applying  $\mathcal{T}$  gives

$$\det(H_{n-1}(F_1)) = \det(H_n(F)), \text{ where } F_1(x) = \frac{1+t}{1+tx^2-x^2F_1(x)}.$$

Applying  $\mathcal{T}$  again, we obtain

$$(1+t)^n \det(H_{n-1}(F_2)) = \det(H_n(F_1)), \text{ where } F_2(x) = \frac{1}{1-tx^2-x^2F_2(x)}.$$

This implies  $F_2 = F$ , and hence the recurrence  $\det(H_n(F)) = (1+t)^{n-1} \det(H_{n-2}(F))$ , with initial condition  $\det(H_1(F)) = 1$ , and  $\det(H_2(F)) = 1+t$ .

**Example 3:** ANOTHER PROOF OF PROPOSITION 2.1. Consider the continued fraction

$$F(x) = \frac{1}{1-tx-xF(x)},$$

where  $F(x)$  is the generating function for the Catalan numbers for  $t=0$  and the large Schröder numbers for  $t=1$ .

Under Proposition 4.2(i) we have a unique  $G_1$  such that  $G_1(x) = F(x)$  and  $\det(H_{n-1}^1(G_1)) = \det(H_n(F))$ . Taking  $G_2 = (G_1(x) - 1)/x = (F(x) - 1)/x$ , we have

$$G_2(x) = \frac{(1+t)}{1-(2+t)x-x^2G_2(x)}$$

where  $\det(H_{n-1}(G_2)) = \det(H_{n-1}^1(F))$  and  $u(x) = (1-(2+t)x)/(1+t)$ .

Under Proposition 4.1 we have a unique  $G_3$

$$G_3(x) = \frac{1}{1-(2+t)x-(1+t)x^2G_3(x)},$$

with  $G_3(x) = G_2/(1+t)$  and  $\det(H_{n-1}(G_3)) = (1+t)^{-(n-1)} \det(H_{n-1}(G_2))$ .

Under Proposition 4.2(ii) we have a unique  $G_4$  such that  $G_4(x) = (1+t)G_3(x)$  and  $\det(H_{n-2}(G_4)) = \det(H_{n-1}(G_3))$ .

We see that  $G_4(x) = G_2(x)$ ; thus  $\det(H_{n-1}(G_2)) = (1+t)^{n-1} \det(H_{n-2}(G_2))$  with  $\det(H_1(G_2)) = 1+t$ . Hence  $\det(H_n(F)) = \det(H_{n-1}(G_2)) = (1+t)^{n(n-1)/2}$ .

**Example 4:** ANOTHER PROOF OF PROPOSITION 2.3. To compute  $\det(H_n^1(F))$ , first we consider

$$H_n^1(F) = H_n(F_1), \text{ where } F_1 = \frac{(t+1)x}{1-(2+t)x^2-x^3F_1}.$$

Applying  $\mathcal{T}$  shows that  $\det(H_n(F_1)) = -(1+t)^n \det(H_{n-2}(F_1))$ .

**Example 5:** For  $\ell = 3$ , recall the functional equation

$$F_0(x) = \frac{1}{1 - tx^3 - x^2 F_0(x)}.$$

For arbitrary  $t$ , our transformation gives more and more complicated expressions. This is not surprising since the Hankel determinants do not factor nicely. However, for  $t = 1$  and for  $k = 1, 2, 3$ , the transformation gives nice results similar to that of Proposition 1.3: indeed, sequences of  $\det(H_n^k(F_0))$  also have period 14. For  $k = 4$  there is an interesting result.

**Subexample 5i:** The sequence for  $\det(H_n^1(F_0))$  starts with  $0, -1, 0, 1, 1, 0, -1, 0, 1, 0, -1, -1, 0, 1$ . If we define  $F_1$  so that  $F_0(x) = 1 + xF_1(x)$ , then

$$\det(H_n(F_1)) = \det(H_n^1(F_0)), \text{ with } F_1 = \frac{x(x+1)}{1 - 2x^2 - x^3 - x^3 F_1} \text{ and } d = 1.$$

Then applying  $\mathcal{T}$  repeatedly so  $\mathcal{T}(F_i) = F_{i+1}$ , we obtain

$$\begin{aligned} \det(H_{n-2}(F_2)) &= -\det(H_n(F_1)), \text{ where } F_2 = \frac{x}{(x+1)(1-x-x^2-x^3 F_2)} \text{ and } d = 1; \\ \det(H_{n-2}(F_3)) &= -\det(H_n(F_2)), \text{ where } F_3 = \frac{1+x-x^2}{1-2x^2+x^3-x^3 F_3} \text{ and } d = 0; \\ \det(H_{n-1}(F_4)) &= \det(H_n(F_3)), \text{ where } F_3 = \frac{x}{(1+x-x^2)(1-x-x^2 F_3)} \text{ and } d = 1; \\ \det(H_{n-2}(F_5)) &= -\det(H_n(F_4)), \text{ where } F_5 = \frac{x(x+1)}{1-2x^2-x^3-x^3 F_5}. \end{aligned}$$

The periodicity is established by noticing that  $F_5 = F_1$  and  $\det(H_{n-7}(F_5)) = -\det(H_n(F_1))$ .

**Subexample 5ii:** The sequence for  $\det(H_n^2(F_0))$  starts with  $1, 1, 1, 1, 0, 0, -1, -1, -1, -1, -1, 0, 0, 1$ . If we define  $G_0$  so that  $F_0(x) = 1 + x^2 G_0(x)$ , then  $\det(H_n(G_0)) = \det(H_n^2(F_0))$ ,

$$G_0 = \frac{1+x}{1-2x^2-x^3-x^4 G_0}.$$

One can establish the periodicity using Proposition 4.2. However, this generating function has appeared in Transformation 3 of section 3, where one can see that

$$(4.2) \quad \det(H_n(G_0)) = -\det(H_{n+5}(F_0)).$$

**Subexample 5iii:** The sequence for  $\det(H_n^3(F_0))$  starts with  $1, -1, -1, 0, 0, 0, -1, -1, 1, 1, 0, 0, 0, 1$  and continues with period 14. The verification for this case uses Proposition 4.2(ii) occasionally interspersed with Proposition 4.1. Here we will only sketch the verification. By defining  $F_1$  so that  $F_0(x) = 1 + x^2 + x^3 F_1(x)$ , one finds that

$$F_1 = \frac{1+2x+x^2+x^3}{1-2x^2-x^3-2x^4-x^5 F_1}.$$

For the first transformation, with  $F_2 = \mathcal{T}F_1$ , we find

$$F_2 = \frac{1-2x+x^3}{-1+4x^2+x^3+2x^4-x^2(1+2x+x^2+x^3) F_2},$$

in which  $u(x) = (-1+4x^2+x^3+2x^4)/(1-2x+x^3)$ . Now, since  $u(0) = -1$ , one needs to apply Proposition 4.1 for the next transformation. One proceeds until a generating function equal to  $F_1$  appears to establish the periodicity. We remark that  $d = 0$  for each transformation until the final one which uses Proposition 4.2(ii) with  $d = 3$  (This corresponds to a fourth order block).

**Subexample 5iv:** The sequence for  $\det(H_n^4(F_0))$  begins with

$$2, 3, 4, 0, 0, -4, -5, -6, -7, -8, 0, 0, 8, 9, 10, 11, 12, 0, 0, -12, -13, -14, -15, -16, 0, 0, 16, \dots$$

For  $n \geq 8$ , an essence of periodicity can be gleaned from the recurrence

$$\det(H_n^4(F_0)) = 4 \det(H_{n-1}(F_0)) - \det(H_{n-7}^4(F_0)),$$

for which we sketch a proof, often omitting the functional equations.

We will be applying the transformation  $\mathcal{T}$  eight times, alternating its definition to be first under Proposition 4.1 and then under Proposition 4.2(ii). Let  $F_1$  satisfy  $F_0 = 1 + x^2 + x^3 + x^4 F_1$ . Hence,  $\det(H_n(F_1)) = \det(H_n^4(F_0))$ , and

$$F_1 = \frac{2 + 3x + 2x^2 + 2x^3 + x^4}{1 - 2x^2 - x^3 - 2x^4 - 2x^5 - x^6 F_1}.$$

Here  $u(0) = \frac{1}{2}$ , where  $u(x)$  is for  $F_1$ . Thus, with  $F_2 = \mathcal{T}F_1$ ,  $\det(H_n(F_2)) = (\frac{1}{2})^n \det(H_n(F_1))$ . Now  $d = 0$ , where  $d$  is for  $F_2$ . With  $F_3 = \mathcal{T}F_2$ ,  $\det(H_{n-1}(F_3)) = \det(H_n(F_2))$ .

Here  $u(0) = \frac{4}{3}$ , where  $u(x)$  is for  $F_3$ . Thus, with  $F_4 = \mathcal{T}F_3$ ,  $\det(H_{n-1}(F_4)) = (\frac{4}{3})^{n-1} \det(H_{n-1}(F_3))$ . Now  $d = 0$ , where  $d$  is for  $F_4$ . With  $F_5 = \mathcal{T}F_4$ ,  $\det(H_{n-2}(F_5)) = \det(H_{n-1}(F_4))$ .

Here  $u(0) = \frac{9}{8}$ , where  $u(x)$  is for  $F_5$ . Thus, with  $F_6 = \mathcal{T}F_5$ ,  $\det(H_{n-2}(F_6)) = (\frac{9}{8})^{n-2} \det(H_{n-2}(F_5))$ . Now  $d = 0$ , where  $d$  is for  $F_6$ . With  $F_7 = \mathcal{T}F_6$ ,  $\det(H_{n-3}(F_7)) = \det(H_{n-2}(F_6))$ .

Here  $u(0) = \frac{4}{3}$ , where  $u(x)$  is for  $F_7$ . Thus, with  $F_8 = \mathcal{T}F_7$ ,  $\det(H_{n-3}(F_8)) = (\frac{4}{3})^{n-3} \det(H_{n-3}(F_7))$ .

Now  $d = 2$ , where  $d$  is for  $F_8$ . With  $F_9 = \mathcal{T}F_8$ ,  $\det(H_{n-6}(F_9)) = -\det((H_{n-3}(F_8))) = (\frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4}, 0, 0, -\frac{8}{4}, -\frac{9}{4}, -\frac{10}{4}, \dots)$ .

Thus, (surprisingly)

$$(4.3) \quad \begin{aligned} \det(H_{n-6}(F_9)) &= -(\frac{1}{2})^n (\frac{4}{3})^{n-1} (\frac{9}{8})^{n-2} (\frac{4}{3})^{n-3} \det(H_n^4(F_0)) \\ &= -\frac{1}{4} \det(H_n^4(F_0)) \end{aligned}$$

Moreover,

$$F_9 = \frac{20 + 16x - 8x^2 - 4x^3 + x^4}{8(2 - 4x^2 - 2x^3 + x^4) - 16x^4 F_9} = \frac{5}{4} + x + 2x^2 + 3x^3 + 6x^4 + 10x^5 + \dots$$

It is easily verified that  $F_9(x)$  and  $\frac{1}{4} + G_0(x)$ , where  $G_0$  appears in Subexample 5ii, satisfy the same functional equation, and hence are equal. Therefore,

$$\begin{aligned} \det(H_{n-6}(F_9)) &= \left[ \frac{x F_9(x) - y F_9(y)}{x - y} \right]_{n-6} \\ &= \left[ \frac{1}{4} + \frac{x G_0(x) - y G_0(y)}{x - y} \right]_{n-6} \\ &= \frac{1}{4} \det(H_{n-7}^4(F_0)) + \det(H_{n-6}(G_0)) \end{aligned}$$

where  $\frac{1}{4} \det(H_{n-7}^4(F_0))$  is  $\frac{1}{4}$  times the determinant of the 1,1-minor of  $H_{n-6}(G_0)$ , equivalently of  $H_{n-6}^2(F_0)$ . Combining this with identity (4.3) and noting  $\det(H_n(G_0)) = -\det(H_{n+5}(F_0))$  from (4.2) proves the initial recurrence of this subexample.

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DEPARTMENT OF MATHEMATICS, BOISE STATE UNIVERSITY, BOISE, ID, USA 83725  
*E-mail address:* `sulanke@math.boisestate.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY, USA 40506  
*E-mail address:* `gxin@ms.uky.edu`